

ON GAUSSIAN MARGINALS OF UNIFORMLY CONVEX BODIES

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ABSTRACT. Recently, Bo'az Klartag showed that arbitrary convex bodies have Gaussian marginals in most directions. We show that Klartag's quantitative estimates may be improved for many uniformly convex bodies. These include uniformly convex bodies with power type 2, and power type $p > 2$ with some additional type condition. In particular, our results apply to all unit-balls of subspaces of quotients of L_p for $1 < p < \infty$. The same is true when L_p is replaced by S_p^m , the l_p -Schatten class space. We also extend our results to arbitrary uniformly convex bodies with power type p , for $2 \leq p < 4$. These results are obtained by putting the bodies in (surprisingly) non-isotropic positions and by a new concentration of volume observation for uniformly convex bodies.

1. INTRODUCTION

In recent years, numerous results have been obtained of the following nature: let X denote a uniformly distributed vector inside a centrally-symmetric convex body K of volume 1 in \mathbb{R}^n . Let $X_\theta := \langle X, \theta \rangle$ denote its marginal in the direction of $\theta \in S^{n-1}$, where S^{n-1} denotes the Euclidean unit sphere. Show that under suitable conditions on K , the distribution of X_θ is *approximately* Gaussian for *most* directions $\theta \in S^{n-1}$. Of course, the meaning of “approximately” and “most” need to be carefully defined, and vary among the different results.

To better illustrate this, consider the following examples. If $K = [-\frac{1}{2}, \frac{1}{2}]^n$, an n -dimensional cube, and $\theta = \frac{1}{\sqrt{n}}(1, \dots, 1)$, the classical Central Limit Theorem asserts that $\langle X, \theta \rangle$ tends in distribution to a Gaussian with variance $\frac{1}{12}$. Of course this is false for *all* directions $\theta \in S^{n-1}$, as witnessed by the directions aligned with the cube's axes, but does hold for most directions as measured by σ , the Haar probability measure on S^{n-1} . When K is a volume 1 homothetic copy of the Euclidean ball D_n , the fact that (all) marginals are approximately Gaussian is classical, dating back to Maxwell, Poincaré and Borel (see [14] for a historical account). In the broader context of general measures on \mathbb{R}^n with finite second moment, Sudakov [36] showed that most marginals are approximately the same *mixture* of Gaussian distributions. Under some additional conditions on the measure in question, Diaconis and Freedman [13] showed that this mixture can be replaced by a proper Gaussian. A generalized version of both results was given by von Weizsäcker in [40]. Several concrete convex bodies (other than the Euclidean Ball and the Cube), such as the cross-polytope and simplex, were studied in [11].

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Motivated by these and other results, it was conjectured by Antilla, Ball and Perissinaki [1] and Brehm and Voigt [11] (using different and in fact stronger formulations) that all convex bodies in \mathbb{R}^n have at least one marginal which is approximately Gaussian, with the deviation tending to 0 as the dimension n tends to ∞ . This conjecture, referred to as the “Central Limit Problem for Convex Bodies” has been confirmed to hold for various classes of convex bodies ([1],[35], [41],[24],[22]).

Recall that K is called *isotropic* if it has volume 1 and satisfies that $\text{Var}(X_\theta) = L_K^2$ for all $\theta \in S^{n-1}$ and some constant $L_K > 0$, which is called the isotropic constant of K . Here $\text{Var}(Y)$ denotes the variance of the random variable Y . It is well known (e.g. [29]) that every full-dimensional body has an affine image which is isotropic and that this image is unique modulo orthogonal rotations; we will refer to this affine image as the body’s isotropic position. Let us further denote the density function of X_θ by $g_\theta(s) := \text{Vol}(K \cap \{s\theta + \theta^\perp\})$, and let $\phi_\rho(s) := \frac{1}{\sqrt{2\pi\rho}} \exp(-\frac{s^2}{2\rho^2})$ denote the Gaussian density with variance ρ^2 . To emphasize that these notions depend on K , we will usually use $g_\theta(K)$ instead of g_θ , et cet.

Recently, the Central Limit Problem for arbitrary convex bodies was given an affirmative answer by Bo’az Klartag ([20], [21]) in the following sense: for every isotropic convex body in \mathbb{R}^n

$$(1.1) \quad \sigma \{ \theta \in S^{n-1}; d_{TV}(g_\theta(K), \phi_{L_K}) \leq \delta_n \} \geq 1 - \mu_n ,$$

where $d_{TV}(f, g) = \int_{-\infty}^{\infty} |f(s) - g(s)| ds$ is the total-variation metric between the measures given by the densities f, g , and δ_n, μ_n are two series decreasing to 0. Klartag’s results in fact apply to all isotropic log-concave probability measures on \mathbb{R}^n . We refer to [8] for the definition of log-concave measures, and only remark that the Gaussian measure and arbitrary marginals of convex bodies are known to be log-concave. In addition, for suitable $k = k(n)$ increasing with n , the existence of k -dimensional marginals which are approximately Gaussian was also shown. In [20] and later in [16], δ_n and $k(n)$ were shown to have logarithmic dependence in n , and in [21] this was improved to polynomial dependence: there exists some $\kappa_1, \kappa_2 > 0$ such that the results are valid for $\delta_n = n^{-\kappa_1}$ and $k(n) = n^{\kappa_2}$. In addition, it was shown in [21] that one may replace in (1.1) the metric d_{TV} with the stronger notion of proximity d_{Lin}^T , to be defined in (1.5), with $T = L_K n^{\kappa_3}$ and $\delta_n = n^{-\kappa_4}$ for some $\kappa_3, \kappa_4 > 0$. According to [21] and some recent improvement in [19, Section 7], one may use $\kappa_1 = 1/60, \kappa_2 = 1/15, \kappa_3 = 1/24, \kappa_4 = 1/24$ in the above statements.

In this note, which is based on a previous version [27] posted on the arXiv before the announcement of Klartag’s results, we will focus on showing the existence of approximately Gaussian marginals in a strong sense for a rather wide class of symmetric convex bodies. Although our results do not apply to general convex bodies as in Klartag’s work, we are able to obtain better quantitative bounds on the deviation between the body’s marginal and the corresponding Gaussian distribution (the δ_n in (1.1)). Earlier results in this direction which have been most influential to our work include [1], [35] and [22]; other references are given later on. In those and previously mentioned results, approximately Gaussian marginals are found by requiring from K

that its volume be highly concentrated around a thin spherical shell of radius $\sqrt{n}\rho$, for some $\rho > 0$ and $\varepsilon < 1/2$:

$$(1.2) \quad \text{Prob} \left(\left| \frac{|X|}{\sqrt{n}} - \rho \right| \geq \varepsilon \rho \right) \leq \varepsilon .$$

Usually, in order to obtain this type of volume concentration, the body K is put in isotropic position. Following [22] but contrary to other approaches, and perhaps surprisingly, we will see in this note that it turns out to be more useful to put the body K in some non-isotropic *position* (or affine image), for which we can show (1.2). We will say that K is D -sub-isotropic if K satisfies that $\text{Var}(X_\theta) \leq D\rho^2$ for all $\theta \in S^{n-1}$, where $D \geq 1$ is some fixed universal constant.

Let us denote the average density over all possible directions by $g_{avg}(s) := \int_{S^{n-1}} g_\theta(s) d\sigma(\theta)$. Let ρ_θ^2 denote the variance of the distribution corresponding to the density g_θ , and set $\rho_{max} = \max_{\theta \in S^{n-1}} \rho_\theta$ and $\rho_{avg} = \int_{S^{n-1}} \rho_\theta d\sigma(\theta)$. We reserve the symbols $C, C', C_1, C_2, c, c_1, c_2$, etc., to indicate positive universal constants, independent of all other parameters, whose value may change from one appearance to the next.

There are usually two steps in showing the existence of approximately Gaussian marginals: first, show that g_{avg} is close to ϕ_ρ , and then show that most densities g_θ are close to g_{avg} . Again, the meaning of “close to” and “most” vary between the results. In [1], the proximity between two even densities f_1, f_2 was interpreted in a rather weak sense, by using the Kolmogorov metric (for even densities):

$$(1.3) \quad d_{Kol}(f_1, f_2) := \sup_{t>0} \left| \int_{-t}^t f_1(s) ds - \int_{-t}^t f_2(s) ds \right| ,$$

which does not capture the similarity in the tail behaviour of the densities. Note that when comparing a one dimensional log-concave density with a Gaussian one, it is known (see [10, Theorem 3.3]) that d_{TV} and d_{Kol} are equivalent in the sense that $d_{Kol} \leq d_{TV} \leq h(d_{Kol})$ for some function $h(t) = O((t \log(1/t))^{1/2})$. In fact, Klartag obtains some of his results in [20, 21] using d_{Kol} and translates them to d_{TV} using the above remark. Hence all the results stated in this note for d_{Kol} can be easily translated to the total-variation metric.

We summarize the two steps from [1] into a single statement. In fact, our first observation in this note is that the argument of [1], originally derived for an isotropic body, applies to a body in arbitrary position, with some penalty accounting for the deviation from isotropic position, as measured by:

$$C_{iso}(K) := \rho_{max}(K)/\rho_{avg}(K).$$

This more general statement, which was already used (without proof) in [22], reads as follows:

Theorem 1.1 (Generalized from [1]). *Assume that (1.2) holds for a centrally-symmetric convex body K in \mathbb{R}^n . Then for any $\varepsilon < \delta < c_2$:*

$$(1.4) \quad \sigma \{ \theta \in S^{n-1}; d_{Kol}(g_\theta(K), \phi_\rho) \leq \delta \} \geq 1 - C_1 C_{iso}(K) \sqrt{n} \log n \exp \left(- \frac{c_3 n \delta^2}{C_{iso}(K)^2} \right) .$$

Theorem 1.1 is proved in Section 2. We remark that it is easy to check that $c_1 \rho_{avg} \leq \rho \leq c_2 \rho_{avg}$ (for some universal constants $c_1, c_2 > 0$), whenever ρ satisfies (1.2), so we will sometimes use $\rho_{max}(K)/\rho$ in place of the above definition of $C_{iso}(K)$.

In [35], Sasha Sodin interpreted the proximity between two even densities f_1, f_2 in a much stronger sense, by measuring the following Linnik type quantity (see [18]):

$$(1.5) \quad d_{Lin}^T(f_1, f_2) := \sup_{0 \leq s \leq T} \left| \frac{f_1(s)}{f_2(s)} - 1 \right| ,$$

where T may be as large as some power of n . Of course, this stronger notion requires a stronger condition on the concentration of volume inside K :

$$(1.6) \quad \text{Prob} \left(\left| \frac{|X|}{\sqrt{n}} - \rho \right| \geq t\rho \right) \leq A \exp(-Bn^\nu t^\tau) ,$$

for all $0 \leq t \leq 1$ and some $A, B, \nu, \tau > 0$. In that case, we summarize the two steps in [35] into the following single statement. The following formulation, which is not difficult to check, extends Sodin's result, originally formulated for bodies in D -sub-isotropic position (with the dependence on D implicit in the constants), to arbitrary convex bodies (by explicitly stating the dependence on D via the parameter $C_{iso}(K)$).

Theorem 1.2 ([35]). *Let K denote a centrally-symmetric convex body in \mathbb{R}^n and assume that (1.6) holds. Given $0 < \delta < c$ and $\mu > 0$, set:*

$$T = \rho \min \left(\left(\frac{cnC_{iso}(K)^{-2}\delta^4}{\log n + \log \frac{1}{\delta} + \mu} \right)^{\frac{1}{6}}, (c(A, B, \nu, \tau)\delta)^{\gamma/\nu} n^\gamma \right) ,$$

where $\gamma := \nu/(2 \max(\tau, 1))$ and $c(A, B, \nu, \tau)$ explicitly depends on A, B, ν, τ . Then:

$$(1.7) \quad \sigma \{ \theta \in S^{n-1}; d_{Lin}^T(g_\theta(K), \phi_\rho) \leq \delta \} \geq 1 - \exp(-\mu) .$$

The key step in Klartag's results from [21] was the confirmation that (1.6) holds for arbitrary isotropic convex bodies (and more generally, log-concave densities) with $\nu = 0.33$, $\tau = 3.33$, $\rho = L_K$ and universal constants $A, B > 0$. Plugging this into Theorem 1.2, we see that (1.7) holds for arbitrary isotropic convex bodies with $T = L_K n^{\kappa_3}$ and $\delta = n^{-\kappa_4}$, for e.g. $\kappa_3 = 1/24$, $\kappa_4 = 1/24$, as mentioned earlier.

Klartag's approach to the Central Limit Problem for convex bodies, being completely general, cannot exploit any good properties which certain classes of convex bodies possess. Consequently, certain results for concrete classes which preceded Klartag's solution, still give better quantitative bounds. These classes can be roughly divided into two categories.

The first contains convex bodies possessing certain symmetries; these include the l_p^n unit-balls ([1],[35]), more generally arbitrary unit-balls of generalized Orlicz norms ([41]), or other types of symmetries ([24], [25]). In a recent progress in this direction, Klartag has obtained in [19] a Berry-Esseen type result for the marginals of an arbitrary convex body symmetric with respect to reflections about coordinate hyperplanes.

The second category contains classes of uniformly convex bodies under certain restrictions ([1],[35],[22]). With any centrally-symmetric convex $K \subset \mathbb{R}^n$ we associate

a norm $\|\cdot\|_K$ on \mathbb{R}^n . The modulus of convexity of K is defined as the following function for $0 < \varepsilon \leq 2$:

$$(1.8) \quad \delta_K(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\|_K ; \|x\|_K, \|y\|_K \leq 1, \|x-y\|_K \geq \varepsilon \right\}.$$

Note that δ_K is affine invariant, so it does not depend on the position of K . A body K is called uniformly convex if $\delta_K(\varepsilon) > 0$ for every $\varepsilon > 0$. A body K is called “ p -convex with constant α ” (see, e.g. [23, Chapter 1.e]), if for all $0 < \varepsilon \leq 2$,

$$(1.9) \quad \delta_K(\varepsilon) \geq \alpha \varepsilon^p.$$

It is known that in such case p cannot be smaller than 2.

The restriction imposed on p -convex bodies is usually via an upper bound on the diameter of K in isotropic position ([1]) or more generally in sub-isotropic position ([35]). For a 2-convex body K with constant α , this restriction on the diameter in isotropic position was previously removed by Klartag and the author in [22]. This was achieved by using Theorem 1.1, which as remarked above, holds in an arbitrary position. By placing K in Löwner’s minimal diameter position, it was shown that $\text{diam}(K) \leq Cn^{1-\lambda}/\lambda$, where $\lambda > 0$ depends only on α , enabling control of the deviation term $C_{iso}(K)$. In order to apply Theorem 1.1, the required concentration (1.2) was then deduced using a concentration result of M. Gromov and V. Milman [17] for uniformly convex bodies (as in [1],[22]). In order to compare the result from [22] with the results in this note, we provide it below:

Theorem 1.3 ([22]). *Let $K \subset \mathbb{R}^n$ denote a 2-convex body with constant α and volume 1. Assume in addition that it is in Löwner’s minimal diameter position, and denote $\rho = \int_K |x| dx / \sqrt{n}$. Then (1.2) holds with:*

$$\varepsilon = c_1 \sqrt{\log n} \alpha^{-1/2} \lambda^{-1} n^{-\lambda},$$

where $\lambda = \lambda(\alpha) > 0$ depends on α only. In addition, for any $\varepsilon < \delta < c_2$:

$$\sigma \left\{ \theta \in S^{n-1}; d_{Kol}(g_\theta(K), \phi_\rho) \leq \delta \right\} \geq 1 - \exp \left(-c_3 \alpha \lambda^2 n^{2\lambda} \delta^2 \right).$$

Our second observation in this note is that the same argument works for arbitrary p -convex bodies ($p > 2$) which have a small type- s constant for large enough s (see Section 3 for definitions). It is easy to show that such bodies have small diameter in Löwner’s position, and so the usual application of the Gromov–Milman concentration gives the desired result. As for the case $p = 2$, the penalty term $C_{iso}(K)$ needs to be handled in order to apply Theorem 1.1. We postpone the formulation of our general result (Theorem 3.9) until Section 3 and only state the following corollary, pertaining to the unit-balls of subspaces of quotients of two useful classes of normed spaces for $1 < p < \infty$: L_p , the class of L_p -integrable functions on $[0, 1]$, and S_p^m , the Schatten class of m by m complex or real matrices, equipped with the norm $\|A\| = (\text{tr}(AA^*))^{p/2})^{1/p}$.

Theorem 1.4. *Let K denote the unit-ball of an n -dimensional subspace of quotient of L_p or S_p^m for $1 < p < \infty$, and assume it has volume 1. Assume in addition that it*

is in Löwner's minimal diameter position, and denote $\rho = \int_K |x| dx / \sqrt{n}$. Then (1.2) holds with:

$$\varepsilon = c_1 \sqrt{rq} (\log n)^{\frac{1}{\max(p,2)}} n^{-\frac{1}{r}},$$

where $r = \max(p, q)$ and $q = p^* = p/(p-1)$. In addition, for any $\varepsilon < \delta < c_2$:

$$\sigma \{ \theta \in S^{n-1}; d_{Kol}(g_\theta(K), \phi_\rho) \leq \delta \} \geq 1 - n^{\frac{5}{2}} \exp \left(-\frac{c_3}{rq} n^{\frac{2}{r}} \delta^2 \right).$$

With our extended formulation of Theorem 1.2 at hand, we can also give analogous results to those of Theorems 1.3, 1.4 (and 3.9 from Section 3) using the stronger notion of proximity between densities (1.5). Indeed, for p -convex bodies as above, the Gromov–Milman argument already implies the stronger concentration assumption (1.6), and the penalty of $C_{iso}(K)$ appearing in Theorem 1.2 is handled exactly as for the former notion of proximity. We will only state the analogue of Theorem 1.4, the analogue of Theorem 1.3 (and 3.9) is stated in Section 3.

Theorem 1.5. *With the same assumptions and notations as in Theorem 1.4, (1.6) holds with:*

$$\nu = \min(2/q, 1), \tau = \max(p, 2), A = 4, B = q^{-2}(cp)^{-p/2}.$$

In addition, (1.7) holds for any $0 < \delta < c$ and $\mu > 0$ with:

$$T = \rho \min \left(\left(\frac{c\delta^4(rq)^{-1}}{\log n + \log \frac{1}{\delta} + \mu} \right)^{\frac{1}{6}} n^{\frac{1}{3r}}, c(p)\delta^{\frac{1}{\max(p,2)}} n^{\frac{1}{2r}} \right).$$

In Section 4, we take on a different approach, which relies on the results of Bobkov and Ledoux from [6]. Contrary to other methods, which need to control the *global* Lipschitz constant of the Euclidean norm $|x|$ w.r.t. $\|\cdot\|_K$, the results in [6] enable us to average out the *local* Lipschitz constant of $|x|$ on K . Unfortunately, our estimate for this average enables us to deduce a result for p -convex bodies only in the range $2 \leq p < 4$. Surprisingly, the position of K which we use to obtain our bounds is “half” way between the isotropic and the minimal mean-width positions (see Theorem 4.6). We state the result only using the stronger notion of proximity d_{Lin}^T , an analogous version using the weaker d_{Kol} metric may also be deduced.

Theorem 1.6. *Let $K \subset \mathbb{R}^n$ denote a p -convex body with constant α for $2 \leq p < 4$, and assume it has volume 1. Assume in addition that it is in the position given by Theorem 4.6 below, and denote $\rho^2 = \int_K |x|^2 dx / n$. Then (1.6) holds with*

$$\nu = \frac{3}{8} - \frac{1}{2q}, \tau = \frac{1}{2}, A = 2, B = c\alpha^{\frac{1}{2p}} / \min(f(p, \alpha), \log(1+n))^{\frac{1}{2}},$$

where $q = p^* = p/(p-1)$ and f is some implicit function (given by Lemma 4.5). In addition, (1.7) holds for any $0 < \delta < c$ and $\mu > 0$ with:

$$T = \rho \min \left(\left(\frac{c\alpha^{\frac{1}{p}} \min(f(p, \alpha), \log(1+n))^{-1} \delta^4}{\log n + \log \frac{1}{\delta} + \mu} \right)^{\frac{1}{6}} n^{\frac{1}{6p}}, (c(p, \alpha)\delta)^{\frac{1}{2}} n^{\frac{3}{16} - \frac{1}{4q}} \right).$$

Note that for the range $2 \leq p < 4$, the latter Theorem holds without *any* assumptions on the diameter of the p -convex body (or the type-constant of the corresponding space). Even for $p = 2$, this is an improvement over Theorem 1.3 which was proved in [22] and Theorem 3.6, since there an implicit function $\lambda = \lambda(\alpha)$ appears in several expressions and in particular in the exponent of n (in Theorem 1.6 we can always replace f by $\log(1 + n)$).

As a corollary, we strengthen Theorem 1.4 for unit-balls of subspaces of quotients of L_p or S_p^m with $1 < p < \frac{16}{13}$, since in this range, r in Theorem 1.4 exceeds the value of $\frac{16}{3}$. These bodies are known to be 2-convex with constant $\alpha = c(p - 1)$ (see Lemma 3.10), so we may apply Theorem 1.6.

Corollary 1.7. *Let K be the unit-ball of an n -dimensional subspace of quotient of L_p or S_p^m for $1 < p \leq \frac{16}{13}$, and assume it has volume 1. Assume in addition that it is in the position given by Theorem 4.6 below, and denote $\rho^2 = \int_K |x|^2 dx/n$. Then (1.6) holds with:*

$$\nu = \frac{1}{8}, \tau = \frac{1}{2}, A = 2, B = c(p - 1)^{\frac{1}{4}} / \log(1 + n)^{\frac{1}{2}}.$$

In addition, (1.7) holds for any $0 < \delta < c$ and $\mu > 0$ with:

$$T = \rho \min \left(\left(\frac{c\alpha^{\frac{1}{2}} \log(1 + n)^{-1} \delta^4}{\log n + \log \frac{1}{\delta} + \mu} \right)^{\frac{1}{6}} n^{\frac{1}{12}}, (c(p)\delta)^{\frac{1}{2}} n^{\frac{1}{16}} \right).$$

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2. GAUSSIAN MARGINALS IN ARBITRARY POSITION

We dedicate this section to the proof of Theorem 1.1, which was already used in [22] to deduce Theorem 1.3, and which will be used in the next section for proving Theorems 3.9 and 1.4.

Proof of Theorem 1.1. We follow the proof in [1], emphasizing the necessary changes. Denote $G(t) = \int_{S^{n-1}} \int_{-t}^t g_\theta(s) ds d\sigma(\theta)$ and $\Phi_\rho(t) = \int_{-t}^t \phi_\rho(s) ds$. It was shown in [1] that under the condition (1.3):

$$(2.1) \quad |G(t) - \Phi_\rho(t)| \leq 4\varepsilon + \frac{c}{\sqrt{n}}$$

for any $t > 0$, and this is still valid for any position of K since the isotropicity of K was not used in the argument at all. Another important observation from [1], which holds regardless of position, is that for every $t > 0$, $\int_{-t}^t g_\theta(s) ds$ is a reciprocal of a

norm. More precisely, denoting:

$$\|x\|_t = \frac{|x|}{\int_{-t}^t g_{\frac{x}{|x|}}(s) ds},$$

it was shown in [1] that $\|\cdot\|_t$ is a norm for any $t > 0$ and that:

$$(2.2) \quad a_t\left(\frac{x}{|x|}\right)|x| \leq \|x\|_t \leq b_t\left(\frac{x}{|x|}\right)|x|,$$

where a_t, b_t satisfy for $\theta \in S^{n-1}$:

$$(2.3) \quad a_t(\theta) = c_1 \max\left(\frac{\rho_\theta}{t}, 1\right), \quad b_t(\theta) = c_2 \max\left(\frac{\rho_\theta}{t}, 1\right).$$

To conclude that given $t > 0$, the individual marginals $\int_{-t}^t g_\theta(s) ds$ are close to their spherical mean $G(t)$ (which is already known to be close to $\Phi_\rho(t)$), the authors of [1] invoke a classical result on concentration of Lipschitz functions around their mean: if $f : S^{n-1} \rightarrow \mathbb{R}$ is a λ -Lipschitz function then:

$$(2.4) \quad \sigma \left\{ \theta \in S^{n-1}; f(\theta) - \int_{S^{n-1}} f(\xi) d\sigma(\xi) \geq \delta \right\} \leq \exp(-Cn\delta^2/\lambda^2).$$

To this end, an estimate on the Lipschitz constant of $\int_{-t}^t g_\theta(s) ds$ is needed. Unfortunately, a straightforward application of the argument in [1] (as reproduced below) yields a Lipschitz constant of $C \frac{\rho_{max}}{\rho_{min}}$, where $\rho_{min} = \min_{\theta \in S^{n-1}} \rho_\theta$, and this is not good enough for our purposes. We therefore modify the argument a little. For $0 < \gamma < 1$, let:

$$A_\gamma = \{ \theta \in S^{n-1}; \rho_\theta \geq (1 - \gamma)\rho_{avg} \}.$$

Since $\rho_\theta^2 = \int_K \langle x, \theta \rangle^2 dx$, it is clear that ρ_θ is a norm in θ , and therefore its Lipschitz constant is bounded above by ρ_{max} . Hence by (2.4):

$$\sigma(A_\gamma) \geq 1 - \exp\left(-\frac{Cn\gamma^2}{C_{iso}(K)^2}\right).$$

This means that for most directions, we can actually use $(1 - \gamma)\rho_{avg}$ as a lower bound on ρ_θ . Let $a_t^\gamma := c_1 \max((1 - \gamma)\rho_{avg}/t, 1)$, and define the modified norm $\|x\|_t^\gamma := \max(\|x\|_t, a_t^\gamma|x|)$. Note that by (2.2) and (2.3), we did not alter the norm on $\theta \in A_\gamma$, for which $\int_{-t}^t g_\theta(s) ds = 1/\|\theta\|_t^\gamma$. As in [1], we evaluate the Lipschitz constant of the latter expression:

$$\left| \frac{1}{\|\theta_1\|_t^\gamma} - \frac{1}{\|\theta_2\|_t^\gamma} \right| \leq \frac{\|\theta_1 - \theta_2\|_t^\gamma}{\|\theta_1\|_t^\gamma \|\theta_2\|_t^\gamma} \leq \frac{b_t\left(\frac{\theta_1 - \theta_2}{\|\theta_1 - \theta_2\|_t}\right)}{(a_t^\gamma)^2} |\theta_1 - \theta_2| \leq C \frac{C_{iso}(K)}{(1 - \gamma)} |\theta_1 - \theta_2|,$$

regardless of the value of t . Denoting $G^\gamma(t) = \int_{S^{n-1}} \frac{1}{\|\theta\|_t^\gamma} d\sigma(\theta)$, (2.4) implies that:

$$\sigma \left\{ \theta \in S^{n-1}; \left| \frac{1}{\|\theta\|_t^\gamma} - G^\gamma(t) \right| \geq \eta \right\} \leq 2 \exp\left(-\frac{Cn\eta^2(1 - \gamma)^2}{C_{iso}(K)^2}\right).$$

Since $\frac{1}{\|\theta\|_t^\gamma}$ and $\int_{-t}^t g_\theta(s)ds$ are both bounded from above by absolute constants and differ only outside the set A_γ , we have:

$$(2.5) \quad |G^\gamma(t) - G(t)| \leq C'\sigma \{\theta \notin A_\gamma\} \leq C' \exp\left(-\frac{Cn\gamma^2}{C_{iso}(K)^2}\right).$$

We can now conclude as follows. Let $\delta > 0$ be given, and assume that δ is not greater than some absolute constant $c > 0$, so that we may define $\gamma = C_0\delta < 1/2$. The fact that ρ_θ is a norm implies (e.g. [30]) that $\rho_{max} \leq C\sqrt{n}\rho_{avg}$, and therefore choosing C_0 above big enough, we always have by (2.5), $|G^\gamma(t) - G(t)| \leq \delta/2$. Hence:

$$\begin{aligned} & \sigma \left\{ \theta \in S^{n-1}; \left| \int_{-t}^t g_\theta(s)ds - G(t) \right| \geq \delta \text{ or } \theta \notin A_\gamma \right\} \\ & \leq \sigma \{\theta \notin A_\gamma\} + \sigma \left\{ \theta \in S^{n-1}; \left| \frac{1}{\|\theta\|_t^\gamma} - G^\gamma(t) \right| \geq \delta - |G^\gamma(t) - G(t)| \right\} \\ & \leq \exp\left(-\frac{Cn\gamma^2}{C_{iso}(K)^2}\right) + 2 \exp\left(-\frac{Cn(\delta/2)^2(1-\gamma)^2}{C_{iso}(K)^2}\right) \leq 3 \exp\left(-\frac{Cn\delta^2}{C_{iso}(K)^2}\right). \end{aligned}$$

Together with (2.1), and denoting $H_\theta(t) = \left| \int_{-t}^t g_\theta(s)ds - \int_{-t}^t \phi_\rho(s)ds \right|$, we have for each $t > 0$:

$$\sigma \left\{ \theta \in S^{n-1}; H_\theta(t) \geq \delta + 4\varepsilon + \frac{c}{\sqrt{n}} \text{ or } \theta \notin A_\gamma \right\} \leq 3 \exp\left(-\frac{Cn\delta^2}{C_{iso}(K)^2}\right).$$

To pass from this estimate to one which holds for all $t > 0$ simultaneously, we use the same argument as in [1], by “pinning” down $H_\theta(t)$ at $C\sqrt{n}\log(n)C_{iso}(K)$ points evenly spread on the interval $[0, C'\max(\rho, \rho_{max})\log(n)]$. Since by our choice of γ , for $\theta \in A_\gamma$ we have $\rho_\theta \geq \rho_{avg}/2$, it is easy to verify (as in [1]) that the Lipschitz constant of $H_\theta(t)$ w.r.t. t is bounded above by $C/\min(\rho_{avg}, \rho)$ on A_γ . By the remark after Theorem 1.1, we know that ρ and ρ_{avg} are equivalent to within universal constants, so the latter Lipschitz constant is bounded above by C'/ρ_{avg} . Since the distance between two consecutive “pinned” points is $C\rho_{avg}/\sqrt{n}$, this ensures that $H_\theta(t)$ does not change by more than C''/\sqrt{n} between consecutive points, and this additional error is absorbed by the earlier error terms. There is no need to control $H_\theta(t)$ for $t \geq C'\max(\rho, \rho_{max})\log(n)$, since both $\int_t^\infty \phi_\rho(s)ds$ (Gaussian decay) and $\int_t^\infty g_\theta(s)ds$ (log-concavity of g_θ , see Lemma 4 in [1]), are smaller than C/\sqrt{n} in that range, and this is again absorbed by the previous error terms. This concludes the proof. \square

3. CONCENTRATION OF VOLUME IN UNIFORMLY CONVEX BODIES WITH GOOD TYPE

In this section, we extend and strengthen the results from [22] to p -convex bodies with “good” type. Recall that the (Rademacher) type- p constant of a Banach space $(X, \|\cdot\|)$ (for $1 \leq p \leq 2$), denoted $T_p(X)$, is the minimal $T > 0$ for which:

$$\left(\mathbb{E} \left\| \sum_{i=1}^m \varepsilon_i x_i \right\|^2 \right)^{1/2} \leq T \left(\sum_{i=1}^m \|x_i\|^p \right)^{1/p}$$

for any $m \geq 1$ and any $x_1, \dots, x_m \in X$, where $\{\varepsilon_i\}$ are i.i.d. random variables uniformly distributed on $\{-1, 1\}$ and \mathbb{E} denotes expectation.

As explained in the Introduction, the existence of Gaussian marginals may be deduced using Theorems 1.1 or 1.2, once we show that the volume inside K is concentrated around a thin spherical shell, in some controllable position of K . A fundamental observation on the concentration of volume inside uniformly convex bodies was given by Gromov and Milman in [17] (see also [2] for a simple proof and [28] for an isoperimetric version). It states that if K is uniformly convex with modulus of convexity δ_K , and $T \subset K$ with $|T| \geq \frac{1}{2}|K|$, then for any $\varepsilon > 0$:

$$(3.1) \quad \frac{\text{Vol}((T + \varepsilon K) \cap K)}{\text{Vol}(K)} \geq 1 - 2 \exp(-2n\delta_K(\varepsilon)).$$

It is easy to see that the latter is equivalent to the concentration around their mean of functions on K which are Lipschitz w.r.t. $\|\cdot\|_K$.

Despite this attractive property of uniformly convex bodies, it is still a hard task to deduce concentration of volume around some spherical shell. The difficulty lies in the fact that for a convex body K , the function $|x|$ has a Lipschitz constant of $\text{diam}(K)$ w.r.t. $\|\cdot\|_K$, and this may be too big to be of use. In the next section, we describe an approach for which we will only need to control the *average* Lipschitz constant of $|x|$ on K , thereby eliminating the need to control $\text{diam}(K)$. In this section, as in [22], we use (3.1) in a direct manner, by putting K in a position for which we have control over $\text{diam}(K)$. This will be ensured by the type condition on K .

We will use the following lemma, which is easy to deduce from (3.1) and the discussion above (see e.g. [1] or [22, Lemma 5.2]):

Lemma 3.1. *Let $K \subset \mathbb{R}^n$ be a p -convex body with constant α and of volume 1. Then for any 1-Lipschitz (w.r.t. $|\cdot|$) function f on K :*

$$\text{Vol} \left\{ x \in K; \left| f(x) - \int_K f(y) dy \right| \geq \text{diam}(K)t \right\} \leq 4 \exp(-2c^p \alpha n t^p).$$

Denoting $\rho = \int_K |x| dx / \sqrt{n}$ and $R = \text{diam}(K) / \sqrt{n}$, we deduce:

$$(3.2) \quad \text{Vol} \left\{ x \in K; \left| \frac{|x|}{\sqrt{n}} - \rho \right| \geq Rt \right\} \leq 4 \exp(-2c^p \alpha n t^p).$$

We see that in order to get some non-trivial concentration, we need to ensure that $R \ll n^{1/p}$. We will make use of the following lemma from [26] (which appeared first in an equivalent form in [12]):

Lemma 3.2. *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body in Löwner's minimal diameter position. Then:*

$$M_2(K) \text{diam}(K) \leq T_2(X_K),$$

where $M_2(K) = \left(\int_{S^{n-1}} \|\theta\|_K^2 d\sigma(\theta) \right)^{\frac{1}{2}}$.

By Jensen's inequality and polar integration, it is immediate for a body of volume 1 that $M_2(K) \geq C/\sqrt{n}$, hence in Löwner's position $\text{diam}(K) \leq \sqrt{n} T_2(X_K)$. By

the results from [39], it is enough to evaluate the type-2 constant of an n -dimensional Banach space on n vectors, and from this it is easy to see that $T_2(X_K) \leq Cn^{\frac{1}{s}-\frac{1}{2}}T_s(X_K)$. We conclude:

Corollary 3.3. *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body of volume 1 in Löwner's minimal diameter position. Then for any $1 \leq s \leq 2$:*

$$\text{diam}(K) \leq Cn^{\frac{1}{s}}T_s(X_K).$$

Combining this with (3.2), we immediately have:

Proposition 3.4. *Let $K \subset \mathbb{R}^n$ be a p -convex body with constant α and of volume 1. Assume in addition that it is in Löwner's minimal diameter position, and denote $\rho = \int_K |x| dx / \sqrt{n}$. Then for any $1 \leq s \leq 2$ we have:*

$$(3.3) \quad \text{Vol} \left\{ x \in K; \left| \frac{|x|}{\sqrt{n}} - \rho \right| \geq t \right\} \leq 4 \exp \left(-2c^p \alpha n^{1+\frac{p}{2}-\frac{p}{s}} \left(\frac{t}{T_s(X_K)} \right)^p \right).$$

In order to get a meaningful result, i.e. a positive power in the exponent of n , we see that we need to have a bounded type- s constant $T_s(X_K)$ for $s > \frac{2p}{p+2}$. It was shown in [22] that for a 2-convex body K with constant α , this is always satisfied for some $s = s(\alpha) > 1$. More precisely, using the same notations as in [22], it was shown that there exists a $0 < \lambda < 1/2$ depending solely on α , such that for $s = \frac{1}{1-\lambda}$ we have $T_s(X_K) \leq 1/\lambda$. By Corollary 3.3, this means that a 2-convex body K with constant α , having volume 1 and in Löwner's position, always satisfies:

$$(3.4) \quad \text{diam}(K) \leq Cn^{1-\lambda}/\lambda.$$

Plugging this into (3.3), we see that for such a body:

$$(3.5) \quad \text{Vol} \left\{ x \in K; \left| \frac{|x|}{\sqrt{n}} - \rho \right| \geq t \right\} \leq 4 \exp \left(-c\alpha n^{2\lambda} \lambda^2 t^2 \right).$$

Since in any position (e.g. [29]):

$$(3.6) \quad \rho \geq c_1 L_K \geq c_2,$$

we get exactly the spherical concentration condition (1.6) needed for applying Theorem 1.2. It remains to evaluate $C_{iso}(K)$, appearing in Theorem 1.2. We argue as in [22], that $\rho_{max}(K)$ may be evaluated just by examining the radii of the circumscribing ball of K and the inscribed Euclidean ball of $\tilde{K} = T(K)$, where T is a volume preserving linear transformation such that \tilde{K} is isotropic. Indeed, it is clear that $\rho_{max} = \|T^{-1}\|_{op} L_K$, where $\|\cdot\|_{op}$ denotes the operator norm. And if $K \subset R D_n$ and $\tilde{K} \supset r D_n$, where D_n denotes the Euclidean unit ball, it is clear that $\|T^{-1}\|_{op} \leq R/r$. In order to evaluate the radius of the inscribed ball of \tilde{K} , we recall the following result from [22]:

Lemma 3.5 ([22]). *Let $K \subset \mathbb{R}^n$ denote a 2-convex body with constant α and volume 1. If K is in isotropic position then:*

$$(3.7) \quad c\sqrt{\alpha n} L_K D_n \subset K,$$

implying in particular that $L_K \leq C/\sqrt{\alpha}$.

Using (3.4) and Lemma 3.5, we deduce that $\rho_{\max}(K) \leq Cn^{\frac{1}{2}-\lambda}\alpha^{-1/2}\lambda^{-1}$. Using (3.6) and the remark after Theorem 1.1, we conclude that:

$$C_{\text{iso}}(K) \leq Cn^{\frac{1}{2}-\lambda}\alpha^{-1/2}\lambda^{-1}.$$

Plugging everything into Theorem 1.2, we deduce:

Theorem 3.6. *Let $K \subset \mathbb{R}^n$ denote a 2-convex body with constant α and volume 1. Assume in addition that K is in L\"owner's minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. Then (1.6) holds with:*

$$\nu = 2\lambda, \tau = 2, A = 4, B = c\alpha\lambda^2,$$

where $\lambda = \lambda(\alpha) > 0$ depends on α only. In addition, (1.7) holds for any $0 < \delta < c$ and $\mu > 0$ with:

$$T = \rho \min \left(\left(\frac{c\alpha\lambda^2\delta^4}{\log n + \log \frac{1}{\delta} + \mu} \right)^{\frac{1}{6}} n^{\lambda/3}, (c(\alpha)\delta)^{\frac{1}{4}} n^{\lambda/2} \right).$$

We remark that Theorem 1.3 was deduced in [22] by choosing $t = c\sqrt{\log(n)}n^{-\lambda}\lambda^{-1}$ in (3.5) and applying Theorem 1.1.

For $p > 2$ the situation is different, because $\frac{2p}{p+2} > 1$ and we cannot in general guarantee that given p and α , $T_s(X_K)$ is bounded even for $s = \frac{2p}{p+2}$. We will therefore need to additionally impose some requirement on $T_s(X_K)$ for $s > \frac{2p}{p+2}$. Once this is done, we deduce from (3.3), as for the case $p = 2$, the spherical concentration condition (1.6) needed for applying Theorem 1.2. In order to control the term $C_{\text{iso}}(K)$ in this case, we need to generalize Lemma 3.5 to the case of p -convex bodies. It is a mere exercise to repeat the proof in [22], which gives:

Lemma 3.7. *Let $K \subset \mathbb{R}^n$ denote a p -convex body with constant α and volume 1. If K is in isotropic position then:*

$$(3.8) \quad c(\alpha n)^{\frac{1}{p}} L_K D_n \subset K,$$

implying in particular that $L_K \leq Cn^{\frac{1}{2}-\frac{1}{p}}\alpha^{-\frac{1}{p}}$.

Arguing as above, this gives together with Corollary 3.3:

$$C_{\text{iso}}(K) \leq Cn^{\frac{1}{s}-\frac{1}{p}}\alpha^{-\frac{1}{p}}T_s(X_K).$$

Plugging this together with Proposition 3.4 into Theorem 1.2, we deduce:

Theorem 3.8. *Let $K \subset \mathbb{R}^n$ denote a p -convex body with constant α and volume 1. Assume in addition that K is in L\"owner's minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. Then (1.6) holds for any $\frac{2p}{p+2} < s \leq 2$ with:*

$$\nu = 1 + p/2 - p/s, \tau = p, A = 4, B = \alpha(c/T_s(X_K))^p.$$

In addition, (1.7) holds for any $0 < \delta < c$ and $\mu > 0$ with:

$$T = \rho \min \left(\left(\frac{c\alpha^{\frac{2}{p}} T_s(X_K)^{-2} \delta^4}{\log n + \log \frac{1}{\delta} + \mu} \right)^{\frac{1}{6}} n^{\frac{1}{6} + \frac{1}{3p} - \frac{1}{3s}}, (c(p, \alpha, s)\delta)^{\frac{1}{2p}} n^{\frac{1}{4} + \frac{1}{2p} - \frac{1}{2s}} \right).$$

Choosing:

$$t = \frac{\log(n)^{1/p} T_s(X_K)}{c\alpha^{1/p} n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{s}}},$$

we deduce from (3.3) the spherical concentration condition (1.2) needed for applying Theorem 1.1, and conclude:

Theorem 3.9. *Let $K \subset \mathbb{R}^n$ denote a p -convex body with constant α and volume 1. Assume in addition that K is in Löwner's minimal diameter position, and denote $\rho = \int_K |x| dx / \sqrt{n}$. Then (1.2) holds for any $\frac{2p}{p+2} < s \leq 2$ with:*

$$\varepsilon_s = c_1 T_s(X_K) (\log n)^{\frac{1}{p}} \alpha^{-\frac{1}{p}} n^{-(\frac{1}{2} + \frac{1}{p} - \frac{1}{s})}.$$

In addition, for any $\varepsilon_s < \delta < c_2$:

$$\sigma \left\{ \theta \in S^{n-1}; d_{Kol}(g_\theta(K), \phi_\rho) \leq \delta \right\} \geq 1 - n^{\frac{5}{2}} \exp \left(-\frac{c_3 n^{1 + \frac{2}{p} - \frac{2}{s}} \delta^2 \alpha^{\frac{2}{p}}}{T_s(X_K)^2} \right).$$

It remains to deduce Theorems 1.4 and 1.5 about unit-balls of subspaces of quotients of L_p and S_p^m for $1 < p < \infty$. With Theorems 3.8 and 3.9 at hand, we only need to evaluate these bodies' r -convexity and type- s constants, for appropriately chosen r and s . This is done in the following (essentially standard) lemma:

Lemma 3.10. *Let $K \subset \mathbb{R}^n$ denote the unit-ball of a subspace of quotient of L_p or S_p^m , for $1 < p < \infty$. Let $r = \max(p, 2)$, $s = \min(p, 2)$ and $q = p^*$. Then:*

- (1) K is r -convex with constant $\alpha(p) = C \min(p-1, p^{-1} 2^{-p})$.
- (2) $T_s(X_K) \leq C \max(\sqrt{p}, \sqrt{q})$.

Sketch of Proof. We will sketch the proof of the L_p case. The proof of the S_p^m case is exactly the same, since by the results of N. Tomczak-Jaegermann [38], these two classes have equivalent type, cotype and modulus of convexity (up to universal constants), and our proof of the L_p case will only depend on estimates for these parameters.

It is known (e.g. [23, Chapter 1.e]) that up to universal constants, L_p has the same modulus of convexity as l_p , and that the latter space is r -convex with constant $\alpha(p)$. By definition, this is passed on to any subspace of L_p , and it is easy to see that the same holds for any quotient space (by passing to the dual and using the modulus of smoothness, see [22, Lemma 3.4]). Item (1) is thus shown.

To show item (2), first consider the case $p \geq 2$. Since L_q is 2-convex with constant $q-1$, the dual L_p is 2-smooth (see [23, Chapter 1.e] or [22]) with constant $\beta = c(q-1)^{-1} \leq Cp$, and by the above discussion, the same is true for K as a unit-ball of a subspace of quotient of L_p . It is standard (e.g. [22, Lemma 4.3]) that this implies that $T_2(X_K) \leq C\sqrt{\beta} \leq C'\sqrt{p}$. When $p < 2$, we use a different argument. Denote by

$C_q(X)$ the cotype- q constant of a Banach space X and by $\|Rad(X)\|$ the norm of the Rademacher projection on $L_2(X_K)$ (see e.g. [30] for definitions). Assuming that K is the unit-ball of a subspace S of a quotient Q of L_p , we have:

$$T_p(X_K) = T_p(S) \leq T_p(Q) \leq C \|Rad(Q)\| C_q(Q^*),$$

where the first inequality is immediate since type passes to subspaces, and the second one is known (e.g. [30]). But by duality, Q^* is a subspace of L_q , and therefore inherits the cotype- q constant of L_q , which is a universal constant (e.g. [30]). We conclude that $T_p(X_K) \leq C \|Rad(Q)\|$. But again by duality $\|Rad(Q)\| = \|Rad(Q^*)\| \leq \|Rad(L_q)\|$, since Q^* is a subspace of L_q . We use the standard estimates $\|Rad(L_q)\| \leq T_2(L_q) \leq C\sqrt{q}$ (e.g. [22]) to deduce that $T_p(X_K) \leq C\sqrt{q}$. This concludes the proof. \square

Plugging this lemma into Theorems 3.8 and 3.9, Theorems 1.4 and 1.5 are deduced.

4. CONCENTRATION OF VOLUME IN p -CONVEX BODIES FOR $p < 4$

Let K denote a p -convex body in \mathbb{R}^n . As already mentioned, it was first noticed by Gromov and Milman ([17]) that functions on K which are Lipschitz w.r.t. $\|\cdot\|_K$ are in fact concentrated around their mean. This phenomenon has since been further developed by many authors (e.g. [33],[34],[2]). A common property to all of these approaches is that the level of concentration depends on the global Lipschitz constant of the function in question, even if in most places the function has a much smaller local Lipschitz constant. The starting point in the following discussion is the interesting results of Bobkov and Ledoux in [6], which overcome the above mentioned drawback.

Recall that the *entropy* of a non-negative function f w.r.t. a probability measure μ , is defined as:

$$Ent_\mu(f) := \int f \log(f) d\mu - \int f d\mu \log\left(\int f d\mu\right).$$

The expectation and variance of f w.r.t. μ are of-course:

$$E_\mu(f) := \int f d\mu, \quad \text{Var}_\mu(f) := E_\mu((f - E_\mu(f))^2).$$

We will also use the following notation for $q > 0$:

$$\text{Var}_\mu^q(f) := E_\mu(|f - E_\mu(f)|^q).$$

We will use $Ent_K(f)$, $\text{Var}_K(f)$ etc. when the underlying distribution μ is the uniform distribution on K . We also denote by $\|\cdot\|^*$ the dual norm to $\|\cdot\|$, defined as $\|x\|^* = \sup\{|\langle x, y \rangle|; \|y\| \leq 1\}$. The following log-Sobolev type inequality was proved in [6, Proposition 5.4] (we correct here a small misprint which appeared in the original formulation):

Theorem 4.1 ([6]). *Let K be a p -convex body with constant α and volume 1, and let $q = p^* = p/(p-1)$. Then for any smooth function f on K :*

$$(4.1) \quad Ent_K(|f|^q) \leq \frac{2^q}{\Gamma(\frac{n}{p} + 1)^{q/n}} \left(\frac{q}{\alpha}\right)^{q-1} \int_K (\|\nabla f\|_K^*)^q dx.$$

When $p = q = 2$, it is classical that this log-Sobolev type inequality implies a Poincaré type inequality. Indeed, by applying Theorem 4.1 to $f = 1 + \varepsilon g$ and letting ε tend to 0, we immediately have:

$$\mathrm{Var}_K(g) \leq \frac{C}{\alpha n} \int_K (\|\nabla g\|_K^*)^2 dx .$$

More generally, it was shown in [7] that for any $q \leq 2$ and norm $\|\cdot\|$, a q -log-Sobolev type inequality:

$$\forall f \quad \mathrm{Ent}_\mu(|f|^q) \leq C \int \|\nabla f\|^q d\mu ,$$

always implies a q -Poincaré type inequality:

$$\forall f \quad \mathrm{Var}_\mu^q(f) \leq C \frac{2^q}{\log 2} \int \|\nabla f\|^q d\mu .$$

Although with this approach the additional term $\frac{2^q}{\log 2}$ may not be optimal (as in the classical $q = 2$ case), universal constants do not play a role in our discussion. Applying this observation to the q -log-Sobolev inequality in Theorem 4.1 we deduce:

Corollary 4.2. *With the same notations as in Theorem 4.1:*

$$(4.2) \quad \mathrm{Var}_K^q(f) \leq \frac{C}{(\alpha n)^{q-1}} \int_K (\|\nabla f\|_K^*)^q dx .$$

Our goal will be to show some non-trivial concentration of the function $g = |x|^2$ around its mean, which is tantamount to the concentration of volume inside K around a thin spherical shell. The advantage of the estimates in Theorem 4.1 and Corollary 4.2 is that they “average out” the local Lipschitz constant of f (w.r.t. $\|\cdot\|_K$) at $x \in K$, which is precisely $\|\nabla f(x)\|_K^*$. The usual way to deduce exponential concentration of g around its mean is via the Herbst argument, by applying Theorem 4.1 to the function $f = \exp(\lambda g/q)$ (see [6] or [7]) and optimizing over λ . Unfortunately, estimating the right-hand side of (4.1) for the function $\exp(\lambda |x|^2/q)$ is a difficult task. An alternative way, which will a-priori only produces polynomial concentration of g around its mean, is to apply Corollary 4.2 to the function $f = g$ and use Markov’s inequality, in hope that estimating the right-hand side of (4.2) should be easier for g itself. We will see that this will in fact lead to exponential bounds. We remark that it is possible to do the same with $f = g$ in (4.1) and gain an additional logarithmic factor in the resulting concentration, but we avoid this for simplicity. We therefore start by applying Corollary 4.2 to the function $f = |x|^2$:

$$(4.3) \quad \mathrm{Var}_K^q(|x|^2) \leq \frac{C'}{(\alpha n)^{q-1}} \int_K (\|x\|_K^*)^q dx .$$

In the following Proposition we estimate the right-hand side of (4.3). We denote by $M^*(K)$ half the mean-width of K , i.e. $M^*(K) = \int_{S^{n-1}} \|\theta\|_K^* d\sigma(\theta)$. We also denote by $SL(n)$ the group of volume preserving linear transformations in \mathbb{R}^n .

Proposition 4.3. *Let K be a p -convex body with constant α . Assume that K is isotropic and of volume 1, and set $q = p^* = p/(p-1)$. Then for any $T \in SL(n)$:*

$$(\text{Var}_{T(K)}^q(|x|^2))^{\frac{1}{q}} \leq \frac{C'}{(\alpha n)^{\frac{1}{p}}} n^{3/4} M^*(T^*T(K)) L_K .$$

Proof. Since:

$$\int_{T(K)} (\|x\|_{T(K)}^*)^q dx = \int_K (\|x\|_{T^*T(K)}^*)^q dx ,$$

by (4.3) and a standard Lemma of C. Borell [8] (note that $q \leq 2$):

$$(\text{Var}_{T(K)}^q(|x|^2))^{1/q} \leq \frac{C'}{(\alpha n)^{\frac{1}{p}}} \left(\int_K (\|x\|_{T^*T(K)}^*)^q dx \right)^{1/q} \leq \frac{C''}{(\alpha n)^{\frac{1}{p}}} \int_K \|x\|_{T^*T(K)}^* dx .$$

Let us evaluate the integral on the right. First, notice that the contribution of $\{x \in K \setminus C\sqrt{n}L_K D_n\}$ to this integral is negligible. To show this, we turn for simplicity to a recent result of Grigoris Paouris ([31]), who showed that when K is in isotropic position:

$$\text{Vol}(K \setminus C\sqrt{n}L_K t D_n) \leq \exp(-\sqrt{nt})$$

for all $t \geq 1$, hence:

$$\int_{K \setminus C\sqrt{n}L_K D_n} \|x\|_{T^*T(K)}^* dx \leq \exp(-\sqrt{n}) \text{diam}(T^*T(K)) \text{diam}(K) .$$

Since $\text{diam}(T^*T(K)) \leq C_1 \sqrt{n} M^*(T^*T(K))$ and $\text{diam}(K) \leq C_2 n L_K$, we see that the latter integral is bounded by $M^*(T^*T(K)) \exp(-\sqrt{n}/2)$, which will be absorbed by the estimate on the integral inside $K \cap C\sqrt{n}L_K D_n$. We emphasize that neither the isotropic position nor Paouris' estimate are cardinal here; a similar argument using Borell's standard Ψ_1 -estimate will give a negligible term. Denoting $K' = T^*T(K)$, it remains to evaluate:

$$(4.4) \quad \int_{K \cap C\sqrt{n}L_K D_n} \|x\|_{K'}^* dx .$$

To this end, we apply a result of J. Bourgain ([9]) which uses the celebrated ‘‘Majorizing-Measures Theorem’’ of Fernique-Talagrand (see [37]), to deduce that the latter is bounded by $C' n^{3/4} M^*(K') L_K$. We remark that this is essentially the same argument which yields Bourgain's well known bound on the isotropic constant $L_K \leq C n^{1/4} \log(1+n)$. For completeness, we outline Bourgain's argument. The idea is to write $\|x\|_{K'}^*$ as $\sup_{y \in K'} \langle y, x \rangle$, so (4.4) becomes an expectation on a supremum of a sub-Gaussian process. Let X_H denote a random vector on the probability space Ω_H which is uniformly distributed on $K \cap C\sqrt{n}L_K D_n$, and for $y \in \mathbb{R}^n$ denote $H_y := \langle X_H, y \rangle$. For a real-valued random variable H on a probability space $(\Omega, d\omega)$ and $\alpha > 0$, let $\|H\|_{L_{\Psi_\alpha}(\Omega)}$ be defined as:

$$\|H\|_{L_{\Psi_\alpha}(\Omega)} = \inf \left\{ \lambda > 0; \int_{\Omega} \exp((H(\omega)/\lambda)^\alpha) d\omega \leq 2 \right\} .$$

A standard calculation shows that:

$$\begin{aligned} \|H_y\|_{L_{\Psi_2}(\Omega_H)} &\leq \sqrt{\|H_y\|_{L_{\Psi_1}(\Omega_H)} \|H_y\|_{L_\infty(\Omega_H)}} \\ &\leq C_1 \sqrt{\|H_y\|_{L_2(\Omega_H)} \|H_y\|_{L_\infty(\Omega_H)}} \leq C_2 n^{1/4} L_K |y| . \end{aligned}$$

Denoting $H'_y = H_y/(C_2 n^{1/4} L_K)$, the latter implies that the process $\{H'_y\}$ is sub-Gaussian w.r.t. the Euclidean-metric, and hence by the Majorizing-Measures Theorem:

$$E_{\Omega_H} \sup_{y \in K'} H'_y \leq C E_{\Omega_G} \sup_{y \in K'} G_y ,$$

where $G_y := \langle X_G, y \rangle$ and X_G is a random vector on the probability space Ω_G whose distribution is that of a standard n -dimensional Gaussian. This implies that:

$$E_{\Omega_H} \sup_{y \in K'} H_y \leq C_3 n^{1/4} L_K n^{1/2} M^*(K') ,$$

and a similar bound holds for (4.4), since the volume of $K \cap C\sqrt{n}L_K D_n$ is close to 1. \square

It is easy to check (e.g. [29]) that $E_{T(K)}(|x|^2) = \int_{T(K)} |x|^2 dx \geq nL_K^2$, and therefore any time the bound in Proposition 4.3 is asymptotically smaller than nL_K^2 we can deduce a concentration result for $|x|^2$ on K . Unfortunately, we are unable to do so in the isotropic position, which is perhaps the most natural position for such concentration of volume to occur. For example, when K is a 2-convex isotropic body (with constant α), we cannot say much about $M^*(K)$; to the best of our knowledge, the best upper bound was given in [22], where it was shown that in isotropic position $M^*(K) \leq C(\alpha)n^{3/4}$, which is exactly the critical value we wish to be properly below.

Proposition 4.3 was deliberately formulated in a way which enables us to work around this problem. We will use a $T \in SL(n)$ so that $M^*(T^*T(K))$ is minimal. In order to use Theorem 1.1, we will also need to control $C_{iso}(T(K))$, which amounts (as in the previous section) to controlling $\|T\|_{op}$. For 2-convex bodies, the relations between the isotropic, the John and the minimal mean-width positions, were studied in [22]. Recall that the John position of a convex body K is defined as the (unique modulo orthogonal rotations) position with maximal radius of the inscribed Euclidean ball. We summarize the additional relevant results from [22] in the following:

Lemma 4.4 ([22]). *Let K be a 2-convex body with constant α and volume 1.*

(1) *If K is in minimal mean-width position then:*

$$M^*(K) \leq C\sqrt{n} \min\left(\frac{1}{\sqrt{\alpha}}, \log(1+n)\right) .$$

(2) *In fact, the same estimate on $M^*(K)$ is valid in John's position.*

The latter easily generalizes to the case of general p -convex bodies. We sketch the argument for the following lemma (see [23] for definitions):

Lemma 4.5. *Let K be a p -convex body with constant α and volume 1. If K is in minimal mean-width position then:*

$$M^*(K) \leq \sqrt{n} \min(f(p, \alpha), C \log(1 + n)) ,$$

where f is a function depending solely on p and α .

Sketch of proof. Recall that by the classical result of Figiel and Tomczak-Jaegermann on the l -position ([15]), we have that in the minimal mean-width position, $M^*(K) \leq C\sqrt{n} \|Rad(X_K)\|$ for a convex body K of volume 1, where $\|Rad(X_K)\|$ denotes the norm of the Rademacher projection on $L_2(X_K)$ (see e.g. [30] for definitions). Since K is p -convex with constant α , it is classical ([23, Proposition 1.e.2]) that K° is q -smooth ($q = p^*$) with constant $\beta(\alpha, p)$, and therefore ([3, Theorem A.7]) has type- q , with $T_q(X_K^*)$ depending only on p and α . Pisier showed in [32] that $\|Rad(X)\| = \|Rad(X^*)\|$ may be bounded from above by an (explicit) function of $T_q(X^*)$ when $q > 1$, which shows that $M^*(K) \leq \sqrt{n}f(p, \alpha)$. By another important result of Pisier (e.g. [30]), for an n -dimensional Banach space X one always has $\|Rad(X)\| \leq C \log(1 + n)$, showing that $M^*(K) \leq \sqrt{n}C \log(1 + n)$. \square

Combining Lemmas 3.7 and 4.5 with Proposition 4.3, we get a concentration result for p -convex bodies with $2 \leq p < 4$. The concentration will be for $T(K)$, the position which is “half-way” (in the geometric mean sense) between the isotropic position K and the minimal mean-width position $T^*T(K)$.

Theorem 4.6. *Let K be a p -convex body with constant α for $2 \leq p < 4$. Assume that K is isotropic and of volume 1, and set $q = p^*$. Then there exists a position $T(K)$ with $T \in SL(n)$, such that:*

(1)

$$\|T\|_{op} \leq C \frac{n^{\frac{1}{2q}}}{\alpha^{\frac{1}{2p}} L_K^{\frac{1}{2}}} \min(f(p, \alpha), \log(1 + n))^{\frac{1}{2}} .$$

(2)

$$(4.5) \quad (\text{Var}_{T(K)}^q(|x|^2))^{1/q} \leq C n^{\frac{1}{4} + \frac{1}{q}} \alpha^{-\frac{1}{p}} L_K \min(f(p, \alpha), \log(1 + n)) .$$

(3) Set $\rho^2 = \int_{T(K)} |x|^2 dx / n$. Then:

$$\text{Vol} \left\{ x \in T(K); \left| \frac{|x|}{\sqrt{n}} - \rho \right| \geq t\rho \right\} \leq 2 \exp \left(- \frac{c L_K^{1/2} \alpha^{\frac{1}{2p}} n^{\frac{3}{8} - \frac{1}{2q}} t^{\frac{1}{2}}}{\min(f(p, \alpha), \log(1 + n))^{1/2}} \right) .$$

Proof. Since the isotropic and the minimal mean-width positions are defined up to orthogonal rotations, we may find a positive definite $T \in SL(n)$ so that $T^*T(K)$ is in minimal mean-width position, which by Lemma 4.5 and Proposition 4.3 gives (2). Since $\text{diam}(T^*T(K)) \leq C\sqrt{n}M^*(T^*T(K))$, we also have:

$$(4.6) \quad T^*T(K) \subset Cn \min(f(p, \alpha), \log(1 + n)) D_n .$$

By Lemma 3.7, this means that:

$$\|T^*T\|_{op} \leq C \frac{n^{1 - \frac{1}{p}}}{\alpha^{\frac{1}{p}} L_K} \min(f(p, \alpha), \log(1 + n)) ,$$

which gives (1). To deduce (3), we use the results of Bobkov [5] on the growth of L_r norms of polynomials. Note that the function $g(x) = |x|^2 - n\rho^2$ is a polynomial of degree 2, so by [5, Theorem 1] there exists a universal constant $C > 0$ such that:

$$(4.7) \quad E_{T(K)} \left(\exp \left(\frac{|g|^{1/2}}{C E_{T(K)}(|g|^{1/2})} \right) \right) \leq 2 .$$

Since $E_{T(K)}(|g|^{1/2}) \leq E_{T(K)}(|g|^q)^{\frac{1}{2q}} = \text{Var}_{T(K)}^q(|x|^2)^{\frac{1}{2q}}$, using the Chebyshev-Markov inequality, (4.7) and (4.5), yields:

$$\begin{aligned} & \text{Vol} \left\{ x \in T(K); \left| \frac{|x|}{\sqrt{n}} - \rho \right| \geq t\rho \right\} \leq \text{Vol} \left\{ x \in T(K); |x|^2 - n\rho^2 \geq n\rho^2 t \right\} \\ &= \text{Vol} \left\{ x \in T(K); |g(x)|^{1/2} \geq \sqrt{nt}\rho \right\} \leq 2 \exp \left(- \frac{\sqrt{nt}\rho}{C \text{Var}_{T(K)}^q(|x|^2)^{1/2q}} \right) \\ &\leq 2 \exp \left(- \frac{\rho \alpha^{\frac{1}{2p}} n^{\frac{3}{8} - \frac{1}{2q}} t^{\frac{1}{2}}}{C' L_K^{1/2} \min(f(p, \alpha), \log(1+n))^{1/2}} \right). \end{aligned}$$

(3) immediately follows since always $\rho \geq L_K$ (e.g. [29]). \square

Remark 4.7.

- (1) We see from (2) and (3) that we get a non-trivial concentration when $q > \frac{4}{3}$, i.e. $p < 4$; this is due to the extra $n^{\frac{1}{4}}$ term in Proposition 4.3.
- (2) For 2-convex bodies, we can slightly improve the estimate on $\|T\|_{op}$ by taking $T^*T(K)$ to be in John's position. Indeed, by part (2) of Lemma 4.4, we will have the same estimate on $M^*(T^*T(K))$ as the one used in the proof of Theorem 4.6. The advantage of using John's position is that $T^*T(K) \subset CnD_n$, improving the estimate in (4.6), which was used to derive the bound on $\|T\|_{op}$.

The advantage of this theorem over the previous concentration results for p -convex bodies in [1] or [22] is three-fold. In [1], the concentration was shown under certain assumptions on the diameter of the bodies, which is not satisfied for some bodies (as shown in [22] even for $p = 2$). In [22], this restriction on the diameter was removed for $p = 2$, but the resulting concentration depended on an implicit function $\lambda = \lambda(\alpha)$, which appeared in the exponent of n . In Theorem 4.6 for the case $2 \leq p < 4$, the restrictions on the diameter of the bodies are removed, the dependence of the concentration on α is explicit, and this dependence is not in the exponent in any of the expressions.

Since it is well known that $L_K \geq c$ (e.g. [29]), Theorem 4.6 yields a concentration of the form (1.6) required to apply Theorem 1.2 to $T(K)$. It remains to evaluate $C_{iso}(T(K))$, taking into account the remark after Theorem 1.1. Since $\rho_{avg} \geq c\rho \geq cL_K$, using (1) from Theorem 4.6 and $L_K \geq c$, we have:

$$C_{iso}(T(K)) = \frac{\rho_{max}}{\rho_{avg}} \leq \frac{\|T\|_{op} L_K}{cL_K} \leq C n^{\frac{1}{2q}} \alpha^{-\frac{1}{2p}} \min(f(p, \alpha), \log(1+n))^{\frac{1}{2}} .$$

Plugging everything into Theorem 1.2, we deduce Theorem 1.6. Corollary 1.7 is deduced by using the estimates given in Lemma 3.10.

Remark 4.8. Sasha Sodin has brought to our attention that a recent result of S. Bobkov ([4]) shows that all our concentration results for uniformly convex bodies in fact imply isoperimetric inequalities for these bodies (with respect to the Euclidean norm). In fact, in a recent manuscript by Sodin and the author [28], we prove isoperimetric analogues of the Gromov-Milman Theorem for uniformly convex bodies, which may be used directly to obtain isoperimetric inequalities with respect to the Euclidean norm, by employing the estimates in this note.

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